Definition

A sequence (f_n) of functions on $A \subset \mathbb{R}$ to $\mathbb{R}$ converges uniformly to f on A if for any $\varepsilon > 0$, there exists $N := N(\varepsilon)$ such that for any $n \geq N$ for any $x \in A$, $|f_n(x) - f(x)| < \varepsilon$.

Lemma

A sequence (f_n) of functions on $A \subset \mathbb{R}$ to $\mathbb{R}$ does not converges uniformly to f on A if and only if there exists $\varepsilon_0 > 0$, a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A such that $|f_{n_k}(x_k) - f(x_k)| > \varepsilon_0$ for any k.

Example 1.

$f_n(x) = \frac{x}{n}$, $A = [0, 1]$.

(f_n) converges uniformly to 0 on $[0, 1]$.

Pf: Fix $\varepsilon > 0$. By AP, there exists some N such that $\frac{1}{N} < \varepsilon$. For any $n \geq N$, for any $x \in [0, 1]$,

$$|f_n(x) - 0| = \left|\frac{x}{n}\right| \leq \frac{1}{n} < N.$$

Hence, (f_n) converges uniformly to 0 on $[0, 1]$.

Example 2.

$f_n(x) = \frac{x}{n}$, $A = \mathbb{R}$.

f_n does not converge uniformly to any f on $\mathbb{R}$.

Pf: Since (f_n) converges pointwisely to 0 on $\mathbb{R}$, i.e.,

for any $x \in \mathbb{R}$, $f_n(x) \to 0$ as $n \to \infty$.

If (f_n) converges uniformy to f on $\mathbb{R}$, then

$f \equiv 0$.

Take $\varepsilon_0 = \frac{1}{2}$, $n_k = k$, $x_k = k$

Then $|f_{n_k}(x_k) - 0| = |f_k(k)| = 1 > \frac{1}{2}$.

Hence, (f_n) does not converge uniformly to 0 on $\mathbb{R}$.

Example 3.

$f_n(x) = x^n$, $A = [0, 1]$.

(f_n) does not converge uniformly to any f on $[0, 1]$.

Pf: Since f_n converges pointwisely to f where

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1, \end{cases}$$ it suffices to show

f_n does not converge uniformly to f on $[0, 1]$.

Let $\varepsilon_0 = \frac{1}{4}$, $n_k = k$, $x_k = (\frac{1}{2})^{\frac{1}{k}}$.

Then $|f_{n_k}(x_k) - f(x_k)| = \left|\left((\frac{1}{2})^{\frac{1}{k}}\right)^k - 0\right| = \frac{1}{2} > \frac{1}{4}$.

Hence (f_n) does not converges uniformly to f on $[0, 1]$.

Example 4.

$f_n(x) = x^n(1-x)$, $A = [0, 1]$.

(f_n) converges uniformly to 0 on $[0, 1]$.

Pf: Fix $\varepsilon > 0$.

There exists some $N \in \mathbb{N}$ such that $(1-\varepsilon)^N < \varepsilon$.

For any $n \geq N$, for any $x \in [0,1]$, if $x < 1-\varepsilon$,

then $|f_n(x) - 0| \leq x^n < (1-\varepsilon)^N < \varepsilon$.

If $x > 1-\varepsilon$, then

$$|f_n(x) - 0| \leq 1 - x < 1-(1-\varepsilon) = \varepsilon.$$

Hence, (f_n) converges uniformly to 0 on $[0,1]$.

Proposition

Let $(f_n), (g_n)$ be sequences of bounded functions that converge uniformly to f, g on A.

Then $(f_n g_n)$ converges uniformly to fg on A.

Pf: Step 1:

$\exists$ M such that $|f_n|, |g_n|, |f|, |g| < M$ for any n.

Pf: Suppose $|f_n| < M_n$.

By Cauchy Criterion for uniform convergence,

$\exists$ $N \in \mathbb{N}$ s.t. for any $n \geq N$, for any $x \in A$,

$|f_n(x) - f_N(x)| < 1$ and $|f(x) - f_N(x)| < 1$.

By triangle inequality,

$|f(x)|, |f_n(x)| < 1 + M_N$ for all $n \geq N$, $x \in A$.

Let $M = \max\{M_1, \dots, M_{N-1}, M_N + 1\}$

Then $|f_n|, |f| < M$ for any n.

The same for g.

Step 2:

Fix $\varepsilon > 0$.

Take $N \in \mathbb{N}$ s.t. for any $n \geq N$, $x \in A$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2M}, \quad |g_n(x) - f(x)| < \frac{\varepsilon}{2M}.$$

Then

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &\leq \left|(f_n(x) - f(x))g_n(x) + f(x)(g_n(x) - g(x))\right| \\
&\leq |f_n(x) - f(x)|\,|g_n(x) + |f(x)|\,|g_n(x) - g(x)| \\
&< \frac{\varepsilon}{2M} \cdot M + M \cdot \frac{\varepsilon}{2M} \\
&= \varepsilon.
\end{aligned}$$

Hence, $(f_n g_n)$ converges to fg uniformly on A.